

Stability Analysis of the Hybrid Variable Step Size Block Method for Integrating Stiff Initial Value Problems

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Abstract

In this paper, the stability analysis of the hybrid variable step size block method for integrating stiff initial value problems of ordinary differential equation is established. The scheme adopted a variable step size technique. The new hybrids block method for integrating stiff ODEs developed by Sagir et. al. (2023) are varied by the variable step size ratio (r), $r = 1$, $r = \frac{1}{7}$ and $r = \frac{3}{7}$ to obtained different stable methods. However, the paper focuses on obtaining the stability region of the stable methods. The methods are found to satisfy the entire stability criteria, so the scheme is Zero and A-stable capable of solving stiff initial value problems. Existing Stiff initial value problems are solved using the proposed method and the numerical result obtained using the new method are found to be efficient at certain step size (h). Hence, the new scheme is recommended for the solutions of stiff Initial Value Problem of ODEs.

Keywords: Hybrid, Stiff, Block Method, Zero Stability, A-Stability

Introduction

Consider a system of first order stiff initial value problems (IVPs) of the form:

$$y' = f(x, y) \quad X \in [a, b] \quad y(X_0) = y_0 \quad (1)$$

System (1) can be regarded as stiff if its exact solution contains very fast as well as very slow component [1]. Stiff initial value problems occur in many areas of engineering and physical sciences. They are particularly found in electrical circuits, vibrations, chemical reactions, kinetics, automated control, combustion, theory of fluid mechanics e.t.c. The solution is characterized by the presence of transient and steady state component which restrict the step size of many numerical method except method with A-stable properties [2, 3].

Stiff problem usually deviates from been solved analytically due to its complexities and other phenomena which is found within its solution, the transient and steady state components found in its solution make explicit method complex to handle with a very good results. While, numerical solution is much more easier and obtainable in any form of stiff Initial Value Problems of ODEs. Most of the stiff cases have no analytical solutions at all. Hence, choices are always made to what numerical methods would solve any sort of stiff Initial Value Problems of ODEs. The final achievement is to get a method with a solution that has absolutely minimum scale error and computational time. Backward differentiation formula came to be developed by Curtiss and Hirschfield [4], several extensions of [4] was carry out by many

scholars including Cash [5-6], the block aspect of [4] was formulated by [7]. Different scholars work tremendously in making a BBDF method that can handle stiff IVPs with minimum error and computational time, these can be found in [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20] and [21].

Definition 1.1: A general k-step linear multistep method is defined as

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f_{n+j} \quad (2)$$

Where α_j and β_j are constant and $\alpha_k \neq 0$. α_0 and β_0 cannot both be zero at the same time.

For any k step method, α_k is normalized to one. The method is said to be explicit if $\beta_k = 0$ and implicit if $\beta_k \neq 0$.

Definition 1.2: The first and second characteristics polynomials of the equation (2) are define by

$$\rho(\xi) = \sum_{j=0}^k \alpha_j \xi^j \quad (3)$$

And

$$\sigma(\xi) = \sum_{j=0}^k \beta_j \xi^j \quad (4)$$

Definition 1.3: Let Y_m and F_m be vectors defined by

$$Y_m = [y_n, y_{n+1}, y_{n+2}, \dots, y_{n+r-1}]^T,$$

$$F_m = [f_n, f_{n+1}, f_{n+2}, \dots, f_{n+r-1}]^T \text{ respectively}$$

Then a general k-block, r-point method is a matrix finite difference equation of the form

$$Y_m = \sum_{i=1}^k A_i Y_{m-1} + h \sum_{i=0}^k B_i f_{m-1} \quad (5)$$

Where all A_i 's and B_i 's are properly chosen $r \times r$ matrix coefficients and $m=0,1,2,\dots$ represent the block number, $n=mr$ the first step number in the m-th block and r is the proposed block size.

Definition 1.4: The block method (4) is said to be zero stable if the root $R_{(j,j)} = 1(1)^k$ of the first characteristics polynomial $\rho(R) = \det[\sum_{i=0}^k A_i R^{k-i}] = 0$, $A_0 = -1$, satisfies $|R_j| < 1$. If one of the roots is $+1$, we call this root the principal root of $\rho(R)$.

Methodology

Consequently, we will extend the approach to formulas that has been derived by [22], called the new hybrid block method for integrating stiff ODEs, by obtaining different stable methods with a variable step size ratio $r = 1$, $r = \frac{1}{7}$, and $r = \frac{3}{7}$. These formulas are given by

$$y_{n+\frac{1}{2}} = \frac{9(2r+1)}{16(4r+3)(r+1)r^2(40r^2-6r-7)} y_{n-2} + \frac{9(16r^2+8r+1)}{4r^2(40r^4+114r^3+55r^2-33r-14)(2r+3)} y_{n-1} - \frac{3(64r^4+9r^3+5r^2+12r+1)}{16r^2(40r^2-6r-7)} y_n + \frac{9(32r^2+32r^2+10r+1)}{4(r+1)(40r^2-6r-7)} y_{n+1} - \frac{3(64r^4+96r^3+52r^2+12r+1)}{(8r^2+18r+9)(40r^2-6r-7)} y_{n+\frac{3}{2}} + \frac{64r^4+96r^3+52r^2+12r+1}{16(40r^4+114r^3+55r^2-33r-14)} y_{n+2} - \frac{3(8r^2+6r+1)}{40r^2-6r-7} h f_{n+\frac{1}{2}} \quad (5)$$

$$y_{n+1} = \frac{1}{8(r+1)(4r+3)(4r+1)} y_{n-2} + \frac{1}{r^2(r^2+4r+4)(2r+3)} y_{n-1} - \frac{2r^2+3r+1}{24r^2} y_n + \frac{16(2r^2+3r+1)}{9(r+1)} y_{n+\frac{1}{2}} + \frac{16(2r^2+3r+1)}{3(8r^2+18r+9)} y_{n+\frac{3}{2}} - \frac{50r^3+193r^2+208r+62}{72(r^3+5r^2+8r+4)} y_{n+2} + \frac{2r+1}{12(r+2)} h f_{n+1} \quad (6)$$

$$y_{n+\frac{3}{2}} = \frac{9(4r^2+12r+9)}{16(r+1)(r+1)(80r^3+292r^2+288r+81)(4r+1)r^2} y_{n-2} - \frac{9(16r^2+24r+9)}{4r^2(r+1)(r+2)(80r^3+292r^2+288r+81)} y_{n-1} + \frac{64r^4+288r^3+468r^2+324r+81}{16r^2(40r^2+126r+81)} y_n - \frac{3(64r^4+288r^3+468r^2+324r+81)}{(8r^2+16r+1)(40r^2+126r+81)} y_{n+\frac{1}{2}} + \frac{9(64r^4+288r^3+468r^2+324r+81)}{4(80r^4+372r^3+580r^2+369r+81)} y_{n+1} + \frac{3(64r^4+288r^3+468r^2+324r+81)}{16(r^2+3r+2)(40r^2+126r+81)} y_{n+2} + \frac{3(8r^2+18r+9)}{40r^2+126r+51} hf_{n+\frac{3}{2}} \quad (7)$$

$$y_{n+2} = \frac{9(r^2+4r+4)}{(100r^3+411r^2+500r+186)(8r^2+6r+1)} y_{n-2} + \frac{72(r+1)}{r^2(50r^3+243r^2+376r+186)(2r+1)} y_{n-1} - \frac{3(r^4+6r^3+13r^2+12r+4)}{r^2(25r^2+84r+62)} y_n + \frac{128(r^4+6r^3+13r^2+12r+4)}{(8r^2+6r+1)(25r^2+84r+62)} y_{n+\frac{1}{2}} - \frac{72(r^3+5r^2+8r+4)}{(2r+1)(25r^2+84r+62)} y_{n+1} + \frac{384(r^4+6r^3+13r^2+12r+4)}{200r^4+1122r^3+2233r^2+1872r+62} y_{n+\frac{3}{2}} + \frac{6(r^2+3r+2)}{25r^2+84r+62} hf_{n+2} \quad (8)$$

Hence, (5), (6), (7) (8) is called a new hybrid block method for integrating stiff ordinary differential equations.

From the scheme developed, substituting the variable step size ratio $r = 1$, $r = \frac{1}{7}$, and $r = \frac{3}{7}$ Different stable methods are obtained for the variable step size ratios respectively as:

$$y_{n+\frac{1}{2}} = -\frac{1}{244}y_{n-2} + \frac{5}{72}y_{n-1} - \frac{25}{16}y_n + \frac{25}{8}y_{n+1} - \frac{5}{7}y_{n+\frac{3}{2}} + \frac{25}{288}y_{n+2} - \frac{5}{3}hf_{n+\frac{1}{2}}$$

$$y_{n+1} = -\frac{1}{560}y_{n-2} + \frac{1}{45}y_{n-1} - \frac{1}{4}y_n + \frac{32}{45}y_{n+\frac{1}{2}} + \frac{32}{35}y_{n+\frac{3}{2}} - \frac{19}{48}y_{n+2} + \frac{1}{12}hf_{n+1}$$

$$y_{n+\frac{3}{2}} = \frac{17}{7904}y_{n-2} - \frac{49}{1976}y_{n-1} + \frac{1225}{3952}y_n - \frac{245}{247}y_{n+\frac{1}{2}} + \frac{3675}{1976}y_{n+1} - \frac{1225}{7904}y_{n+2} + \frac{104}{247}hf_{n+\frac{3}{2}}$$

$$y_{n+2} = -\frac{3}{665}y_{n-2} + \frac{16}{285}y_{n-1} - \frac{12}{19}y_n + \frac{512}{285}y_{n+\frac{1}{2}} - \frac{48}{19}y_{n+1} + \frac{1536}{665}y_{n+\frac{3}{2}} + \frac{14}{19}hf_{n+2} \quad (9)$$

$$y_{n+\frac{1}{2}} = \frac{453789}{368000}y_{n-2} - \frac{2033647}{423200}y_{n-1} + \frac{9801}{1840}y_n - \frac{3267}{3680}y_{n+1} + \frac{9801}{66125}y_{n+\frac{3}{2}} - \frac{1089}{73600}y_{n+2} + \frac{99}{115}hf_{n+\frac{1}{2}}$$

$$y_{n+1} = -\frac{16807}{17600}y_{n-2} + \frac{16807}{5175}y_{n-1} - \frac{3}{1}y_n + \frac{128}{99}y_{n+\frac{1}{2}} + \frac{384}{575}y_{n+\frac{3}{2}} - \frac{1217}{4800}y_{n+2} + \frac{1}{20}hf_{n+1}$$

$$y_{n+\frac{3}{2}} = \frac{8890903}{6886528}y_{n-2} - \frac{2100875}{469536}y_{n-1} + \frac{330625}{78256}y_n - \frac{330625}{161403}y_{n+\frac{1}{2}} + \frac{330625}{156512}y_{n+1} - \frac{66125}{626048}y_{n+2} + \frac{1725}{4891}hf_{n+\frac{3}{2}}$$

$$(10)$$

$$y_{n+2} = -\frac{632043}{66405950}y_{n-2} + \frac{1075648}{83973}y_{n-1} - \frac{14400}{1217}y_n + \frac{204800}{40161}y_{n+\frac{1}{2}} - \frac{4800}{1217}y_{n+1} + \frac{73728}{27991}y_{n+\frac{3}{2}} + \frac{240}{1217}hf_{n+2}$$

$$y_{n+\frac{1}{2}} = -\frac{218491}{575520}y_{n-2} - \frac{6067327}{2001240}y_{n-1} + \frac{61009}{5232}y_n - \frac{42237}{4360}y_{n+1} + \frac{61009}{32373}y_{n+\frac{3}{2}} - \frac{61009}{296480}y_{n+2} + \frac{741}{109}hf_{n+\frac{1}{2}}$$

$$y_{n+1} = -\frac{16807}{451440}y_{n-2} + \frac{16807}{70227}y_{n-1} - \frac{65}{108}y_n + \frac{160}{171}y_{n+\frac{1}{2}} + \frac{2080}{2673}y_{n+\frac{3}{2}} - \frac{6535}{208080}y_{n+2} + \frac{13}{204}hf_{n+1}$$

$$y_{n+\frac{3}{2}} = \frac{1361367}{30628000}y_{n-2} - \frac{2033647}{6851000}y_{n-1} + \frac{9801}{12400}y_n - \frac{264627}{191425}y_{n+\frac{1}{2}} + \frac{793881}{403000}y_{n+1} - \frac{264627}{2108000}y_{n+2} + \frac{297}{775}hf_{n+\frac{3}{2}}$$

$$(11)$$

$$y_{n+2} = -\frac{4857223}{40975077}y_{n-2} + \frac{1344560}{1764477}y_{n-1} - \frac{28900}{15081}y_n + \frac{3699200}{1241669}y_{n+\frac{1}{2}} - \frac{208080}{65351}y_{n+1} + \frac{3699200}{1493019}y_{n+\frac{3}{2}} + \frac{1020}{5027}hf_{n+2}$$

Zero Stability Analysis of the Method

In this section, we investigate the Zero stability of the stable methods (9), (10) and (11).

Definition 3.1: A Linear Multistep Method is said to be A-Stable if its Stability region covers the entire negative half plane [17].

Definition 3.2: A Linear Multistep Method is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple [17].

The zero stability properties of the stable methods are determined through application of the standard linear test problem.

$$y' = \lambda y \tag{12}$$

$\lambda < 0$ is λ Complex.

Applying (12) in the method (9), where $f(x, y) = \lambda y$, i.e. the method for the variable step size ratio $r = 1$. The method can be written in a matrix form as;

$$\begin{bmatrix} \frac{1-5h}{3} & \frac{-25}{8} & \frac{5}{7} & \frac{-25}{228} \\ -32 & \frac{1-h}{19} & \frac{-32}{7} & \frac{228}{19} \\ \frac{45}{245} & \frac{12}{-3675} & \frac{35}{1-105h} & \frac{48}{1225} \\ \frac{247}{-512} & \frac{1976}{-1536} & \frac{247}{-1536} & \frac{7904}{1-4h} \\ \frac{285}{665} & \frac{665}{665} & \frac{665}{665} & \frac{19}{19} \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{5}{72} & 0 & \frac{-25}{16} \\ 0 & \frac{1}{45} & 0 & \frac{-1}{4} \\ 0 & \frac{-49}{1976} & 0 & \frac{1225}{3952} \\ 0 & \frac{16}{285} & 0 & \frac{-12}{19} \end{bmatrix} \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \frac{-1}{224} \\ 0 & 0 & 0 & \frac{-1}{560} \\ 0 & 0 & 0 & \frac{15}{7904} \\ 0 & 0 & 0 & \frac{-3}{665} \end{bmatrix} \begin{bmatrix} y_{n-\frac{7}{2}} \\ y_{n-3} \\ y_{n-\frac{5}{2}} \\ y_{n-2} \end{bmatrix}$$

(13)

Where A, B and C can be given as

$$\begin{bmatrix} \frac{1-5h}{3} & \frac{-25}{8} & \frac{5}{7} & \frac{-25}{228} \\ -32 & \frac{1-h}{19} & \frac{-32}{7} & \frac{228}{19} \\ \frac{45}{245} & \frac{12}{-3675} & \frac{35}{1-105h} & \frac{48}{1225} \\ \frac{247}{-512} & \frac{1976}{-1536} & \frac{247}{-1536} & \frac{7904}{1-4h} \\ \frac{285}{665} & \frac{665}{665} & \frac{665}{665} & \frac{19}{19} \end{bmatrix}, \begin{bmatrix} 0 & \frac{5}{72} & 0 & \frac{-25}{16} \\ 0 & \frac{1}{45} & 0 & \frac{-1}{4} \\ 0 & \frac{-49}{1976} & 0 & \frac{1225}{3952} \\ 0 & \frac{16}{285} & 0 & \frac{-12}{19} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & \frac{-1}{224} \\ 0 & 0 & 0 & \frac{-1}{560} \\ 0 & 0 & 0 & \frac{15}{7904} \\ 0 & 0 & 0 & \frac{-3}{665} \end{bmatrix}$$

Respectively

With the aid of a maple software, the characteristic polynomial for the variable step size ratio $r = 1$ can be obtain by the relation

$$\det(At^2-Bt-C) = \frac{-5432344}{633555}t^8 + \frac{199656}{23465}t^7 + \frac{1544}{23465}t^6 + \frac{56}{633555}t^5 + \frac{2305267}{1783340}t^8h + \frac{162555503}{26750100}t^7h + \frac{8991}{9386}t^8h + \frac{6991}{2964}t^7h - \frac{63}{4693}t^8h^4 + \frac{1869}{4693}t^7h^3 - \frac{21533}{1971060}t^6h + \frac{297}{93860}t^6h^2 + \frac{1727}{29565900}t^5h + \frac{27}{9386}t^6h^3 + \frac{1}{93860}t^5h^2$$

Put $h\lambda = h = 0$ in (14)

We have

$$R_1(t, 0) = \frac{-5432344}{633555}t^8 + \frac{199656}{23465}t^7 + \frac{1544}{23465}t^6 + \frac{56}{633555}t^5$$
 (15)

Solving The Polynomials (15) for (t). The following is obtained for the roots of the polynomials.

$$t = 0, 0, 0, 0, 0, -0.0053951501, 0.1777068048, 1$$

Similarly, applying (12) in the method (10), where $f(x, y) = \lambda y$, i.e. the method for the variable step size ratio $r = \frac{1}{7}$. The method can be written in a matrix form as;

$$\begin{bmatrix} \frac{1-99h}{115} & \frac{3267}{3680} & \frac{-9801}{66125} & \frac{1089}{73600} \\ \frac{-128}{99} & \frac{1-h}{20} & \frac{-384}{575} & \frac{1217}{4800} \\ \frac{330625}{161403} & \frac{-330625}{156512} & \frac{1-1725h}{4891} & \frac{66125}{626048} \\ \frac{-204800}{40161} & \frac{4800}{1217} & \frac{-73728}{27991} & \frac{1-240h}{1217} \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-2033647}{423200} & 0 & \frac{9801}{1840} \\ 0 & \frac{16807}{5175} & 0 & \frac{-3}{1} \\ 0 & \frac{-2100875}{469536} & 0 & \frac{330625}{78256} \\ 0 & \frac{1075648}{83973} & 0 & \frac{-14400}{1217} \end{bmatrix}$$

$$\begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \frac{453789}{368000} \\ 0 & 0 & 0 & \frac{-16807}{17600} \\ 0 & 0 & 0 & \frac{8890903}{6886528} \\ 0 & 0 & 0 & \frac{-632043}{27991} \end{bmatrix} \begin{bmatrix} y_{n-\frac{7}{2}} \\ y_{n-3} \\ y_{n-\frac{5}{2}} \\ y_{n-2} \end{bmatrix}$$

(16)

Where A, B and C can be given as

$$\begin{bmatrix} \frac{1-99h}{115} & \frac{3267}{3680} & \frac{-9801}{66125} & \frac{1089}{73600} \\ \frac{-128}{99} & \frac{1-h}{20} & \frac{-384}{575} & \frac{1217}{4800} \\ \frac{330625}{161403} & \frac{-330625}{156512} & \frac{1-1725h}{4891} & \frac{66125}{626048} \\ \frac{-204800}{40161} & \frac{4800}{1217} & \frac{-73728}{27991} & \frac{1-240h}{1217} \end{bmatrix} \begin{bmatrix} 0 & \frac{-2033647}{423200} & 0 & \frac{9801}{1840} \\ 0 & \frac{16807}{5175} & 0 & \frac{-3}{1} \\ 0 & \frac{-2100875}{469536} & 0 & \frac{330625}{78256} \\ 0 & \frac{1075648}{83973} & 0 & \frac{-14400}{1217} \end{bmatrix} \text{ And}$$

$$\begin{bmatrix} 0 & 0 & 0 & \frac{453789}{368000} \\ 0 & 0 & 0 & \frac{-16807}{17600} \\ 0 & 0 & 0 & \frac{8890903}{8890903} \\ 0 & 0 & 0 & \frac{6886528}{-632043} \\ 0 & 0 & 0 & \frac{-632043}{27991} \end{bmatrix} \text{ respectively.}$$

With the aid of a maple software, the characteristics polynomial for

$r = \frac{1}{7}$ Can be obtain with relation

$$\det(At^2 - Bt - C) = \frac{1212800}{1168813}t^8 - \frac{1943084}{2004405}t^7 - \frac{634890}{672151}t^6 + \frac{2013975}{10116075}t^5 - \frac{91901911}{6744603}t^8h\lambda + \frac{83442}{155850}t^8(h\lambda)^2 - \frac{8287809}{524810}t^7h\lambda + \frac{35055}{478096}t^8(h\lambda)^3 + \frac{28674}{17405}t^2(h\lambda)^2 - \frac{954}{96284}t^8(h\lambda)^4 + \frac{5598}{977036}t^7(h\lambda)^3 + \frac{994}{5129286}t^6h\lambda + \frac{62}{2174405}t^6(h\lambda)^2 - \frac{1}{21744050}t^5h\lambda + \frac{210}{209048}t^6(h\lambda)^3$$

Put $h\lambda = h = 0$ in

(17)

We have

$$R_{\frac{1}{7}}(t, 0) = \frac{1212800}{1168813}t^8 - \frac{1943084}{2004405}t^7 - \frac{634890}{672151}t^6 + \frac{2013975}{10116075}t^5$$

(18)

Solving the Polynomials (18) for (t).The following is obtained for the roots of the polynomials.

$$t = 0, 0, 0, 0, 0, -0.0949560872, 0.1194343209, 1$$

Similarly, applying (12) in the method (11), where $f(x, y) = \lambda y$, i.e. the method for the variable step size ratio $r = \frac{3}{7}$. The method can be written in a matrix form as;

$$\begin{bmatrix}
 \frac{1-741h}{109} & \frac{42237}{4360} & \frac{-61009}{32373} & \frac{61009}{296480} \\
 \frac{-160}{171} & \frac{1-13h}{204} & \frac{-2080}{2673} & \frac{65351}{208080} \\
 \frac{264627}{264627} & \frac{-793881}{-793881} & \frac{1-297h}{1-297h} & \frac{264627}{264627} \\
 \frac{191425}{-3699200} & \frac{403000}{208080} & \frac{775}{3699200} & \frac{2108000}{1-1020h} \\
 \frac{1241669}{65351} & \frac{65351}{65351} & \frac{1493019}{1493019} & \frac{5027}{5027}
 \end{bmatrix}
 \begin{bmatrix}
 y_{n+\frac{1}{2}} \\
 y_{n+1} \\
 y_{n+\frac{3}{2}} \\
 y_{n+2}
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 & \frac{-6067327}{2001240} & 0 & \frac{61009}{5232} \\
 0 & \frac{16807}{70227} & 0 & \frac{-65}{108} \\
 0 & \frac{-2033647}{-2033647} & 0 & \frac{9801}{9801} \\
 0 & \frac{68510000}{1344560} & 0 & \frac{12400}{-28900} \\
 0 & \frac{1764477}{1764477} & 0 & \frac{15081}{15081}
 \end{bmatrix}$$

$$\begin{bmatrix}
 y_{n-\frac{3}{2}} \\
 y_{n-1} \\
 y_{n-\frac{1}{2}} \\
 y_n
 \end{bmatrix}
 +
 \begin{bmatrix}
 0 & 0 & 0 & \frac{218491}{575529} \\
 0 & 0 & 0 & \frac{-16807}{451440} \\
 0 & 0 & 0 & \frac{1361367}{30628000} \\
 0 & 0 & 0 & \frac{-4857223}{40975077}
 \end{bmatrix}
 \begin{bmatrix}
 y_{n-\frac{7}{2}} \\
 y_{n-3} \\
 y_{n-\frac{5}{2}} \\
 y_{n-2}
 \end{bmatrix}$$

(19)

Where A, B and C can be given as

$$\begin{bmatrix}
 \frac{1-741h}{109} & \frac{42237}{4360} & \frac{-61009}{32373} & \frac{61009}{296480} \\
 \frac{-160}{171} & \frac{1-13h}{204} & \frac{-2080}{2673} & \frac{65351}{208080} \\
 \frac{264627}{264627} & \frac{-793881}{-793881} & \frac{1-297h}{1-297h} & \frac{264627}{264627} \\
 \frac{191425}{-3699200} & \frac{403000}{208080} & \frac{775}{3699200} & \frac{2108000}{1-1020h} \\
 \frac{1241669}{65351} & \frac{65351}{65351} & \frac{1493019}{1493019} & \frac{5027}{5027}
 \end{bmatrix},
 \begin{bmatrix}
 0 & \frac{-6067327}{2001240} & 0 & \frac{61009}{5232} \\
 0 & \frac{16807}{70227} & 0 & \frac{-65}{108} \\
 0 & \frac{-2033647}{-2033647} & 0 & \frac{9801}{9801} \\
 0 & \frac{68510000}{1344560} & 0 & \frac{12400}{-28900} \\
 0 & \frac{1764477}{1764477} & 0 & \frac{15081}{15081}
 \end{bmatrix}
 \text{ And}$$

$$\begin{bmatrix}
 0 & 0 & 0 & \frac{218491}{575529} \\
 0 & 0 & 0 & \frac{-16807}{451440} \\
 0 & 0 & 0 & \frac{1361367}{30628000} \\
 0 & 0 & 0 & \frac{-4857223}{40975077}
 \end{bmatrix}
 \text{ respectively.}$$

With the aid of a maple software, the characteristics polynomial for

$r = \frac{3}{7}$ Can be obtain with the relation

$$\det(At^2 - Bt - C) = \det(At^2 - Bt - C) = \frac{856800}{454881}t^8 - \frac{283084}{1274405}t^7 - \frac{62489}{3823215}t^6 + \frac{12}{19116075}t^5 - \frac{91901911}{6744603}t^8h\lambda + \frac{83442}{155850}t^8(h\lambda)^2 - \frac{8287809}{524810}t^7h\lambda + \frac{35055}{478096}t^8(h\lambda)^3 + \frac{28674}{17405}t^2(h\lambda)^2 - \frac{954}{96284}t^8(h\lambda)^4 + \frac{5598}{977036}t^7(h\lambda)^3 + \frac{994}{5129286}t^6h\lambda + \frac{62}{2174405}t^6(h\lambda)^2 - \frac{1}{21744050}t^5h\lambda + \frac{210}{209048}t^6(h\lambda)^3 \quad (20)$$

Put $h\lambda = h = 0$ in (20) we have

$$\frac{R_3}{7}$$

$$(t, 0) = \frac{856800}{454881}t^8 - \frac{283084}{1274405}t^7 - \frac{62489}{3823215}t^6 + \frac{12}{19116075}t^5 \quad (21)$$

Solving the Polynomials (21) for (t) .The following is obtained for the roots of the polynomials.

$$t = 0, 0, 0, 0, 0, 0.520143171919, -0.13392995390923, 0.02176446791230$$

The following table is obtained for the roots of the polynomials.

Table 3.1: Zero Stability of the formulae

Step Size Ratio (r)	Roots of the stable methods
$r = 1$	$t = 0, 0, 0, 0, 0, -0.0053951501, 0.1777068048, 1$
$r = \frac{1}{7}$	$t = 0, 0, 0, 0, 0, -0.0949560872, 0.1194343209, 1$
$r = \frac{3}{7}$	$t = 0, 0, 0, 0, 0, 0.520143171919, -0.13392995390923, 0.02176446791230$

4. A - Stability region of the Method

This section focuses on obtaining the stability region for the variable step size ratio $r = 1$, $r = \frac{1}{7}$, and $r = \frac{3}{7}$ of the stable methods obtained. The stability regions of (9), (10) and (11) are determined through the application of the standard linear test problem

$$y' = \lambda y \quad (22)$$

$$\lambda < 0, \lambda \text{ Complex.}$$

The stability region is drawn in the $h\lambda$ plane and hence it takes the value of $h\lambda$. The region of absolute stability for the stable methods of the variable step size ratio $r = 1$, $r = \frac{1}{7}$, and $r = \frac{3}{7}$ are shown in fig. 1, fig. 2 and fig. 3.

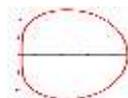


Figure 3: A-stability region of the methods when $r = 1$

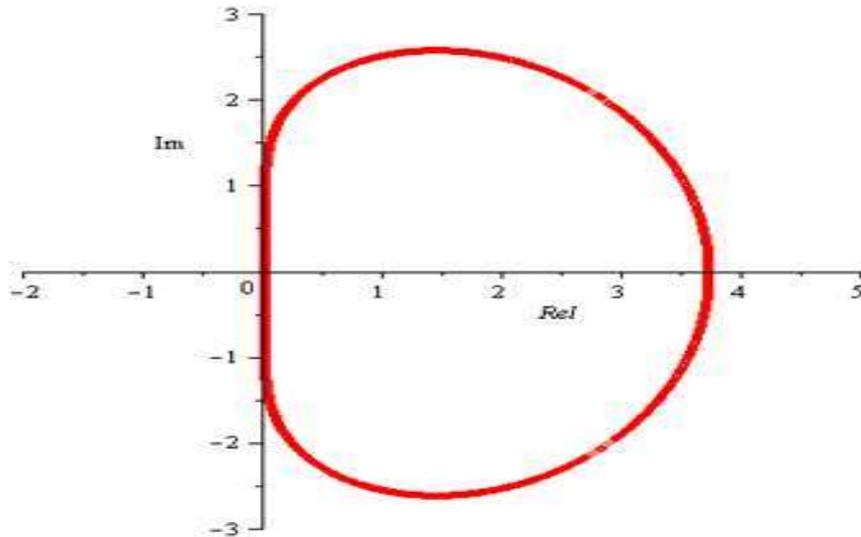


Figure 2: A- Stability Region of the Method when $r = \frac{1}{7}$

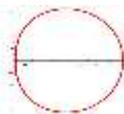


Figure 3: A-stability region of the methods when $r = \frac{3}{7}$

Therefore, it can be concluded that the constructed stability regions for the variable step size ratio $r = 1$, $r = \frac{1}{7}$, and $r = \frac{3}{7}$ covers the entire negative half plane. Therefore the method is suitable for solving stiff problems because of its almost A-stable Property.

5. Test Problems

To validate the Performance of the method, below are some selected Stiff Initial Value Problem of ODEs to consider.

Problem 1: (See [22])

$$\begin{aligned} y' &= 20y_1 - 19y_2 & y_1(0) &= 2 \\ y' &= 19y_1 - 20y_2 & y_2(0) &= 0 \end{aligned} \quad 0 \leq x \leq 20$$

Exact Solution: $y(x) = e^{-5x}$

Problem 2: (See [22])

$$\begin{aligned} y' &= 198y_1 - 199y_2 & y_1(0) &= 1 \\ y' &= 398y_1 - 399y_2 & y_2(0) &= -2 \end{aligned} \quad 0 \leq x \leq 10$$

Exact Solution: $y_1(x) = e^{-x}$, $y_2(x) = -e^{-x}$

Result and Discussions

Some chosen problems were solved using the developed method. The numerical result of the tested problems are put in tables to illustrate the performance of the block methods in solving first order stiff ODEs. The acronyms below are used in the tables.

h = step-size;

MAX-ERR = Maximum Error;

NS= Number of Steps

IVPs = Initial Value Problems

SAHBM = Stability analysis of the hybrid variable step size block method for integrating stiff IVPs.

Table 6.1: Numerical Result for the Test Problem 1

H	Method	Ns	Max-error
10^{-2}	SAHBM ($r = 1$)	100	8.43849 (-5)
	SAHBM ($r = \frac{1}{7}$)	1000	4.48968 (-6)
	SAHBM ($r = \frac{3}{7}$)	1000	4.04864 (-4)
10^{-3}	SAHBM ($r = 1$)	1000	8.45371 (-7)
	SAHBM ($r = \frac{1}{7}$)	10000	4.23669 (-8)
	SAHBM ($r = \frac{3}{7}$)	10000	4.69313 (-6)
10^{-4}	SAHBM ($r = 1$)	10000	8.47282 (-9)
	SAHBM ($r = \frac{1}{7}$)	100000	4.74822 (-10)
	SAHBM ($r = \frac{3}{7}$)	100000	4.82247 (-8)
10^{-5}	SAHBM ($r = 1$)	100000	8.50316 (-11)
	SAHBM ($r = \frac{1}{7}$)	1000000	4.05138 (-12)
	SAHBM ($r = \frac{3}{7}$)	1000000	4.81305 (-10)
10^{-6}	SAHBM ($r = 1$)	1000000	8.52149 (-13)
	SAHBM ($r = \frac{1}{7}$)	10000000	4.82419 (-14)
	SAHBM ($r = \frac{3}{7}$)	10000000	4.28941 (-12)

Table 6.2: Numerical Result for the Test Problem 2

H	Method	Ns	Max-error
10^{-2}	SAHBM ($r = 1$)	100	7.13551 (-5)
	SAHBM ($r = \frac{1}{7}$)	5000	7.00459 (-6)
	SAHBM ($r = \frac{3}{7}$)	5000	7.96425 (-5)
10^{-3}	SAHBM ($r = 1$)	1000	7.32821 (-7)
	SAHBM ($r = \frac{1}{7}$)	50000	7.22318 (-8)
	SAHBM ($r = \frac{3}{7}$)	50000	7.81320 (-7)
10^{-4}	SAHBM ($r = 1$)	10000	7.59457 (-9)
	SAHBM ($r = \frac{1}{7}$)	500000	7.57475 (-11)
	SAHBM ($r = \frac{3}{7}$)	500000	7.05745 (-9)
10^{-5}	SAHBM ($r = 1$)	100000	7.60185 (-11)
	SAHBM ($r = \frac{1}{7}$)	5000000	7.51086 (-13)
	SAHBM ($r = \frac{3}{7}$)	5000000	7.21068 (-11)
10^{-6}	SAHBM ($r = 1$)	1000000	7.89244 (-13)
	SAHBM ($r = \frac{1}{7}$)	50000000	7.24148 (-15)
	SAHBM ($r = \frac{3}{7}$)	50000000	7.34615 (-13)

From table 6.1 and 6.2 the Stability analysis of the hybrid variable step size method is characterized by the decrease in error as the step length h tends to zero for both of $r = 1, r = \frac{1}{7}$ and $r = \frac{1}{7}$. The accuracy also improves as the step length is reduced.

Similarly, the solution at any fixed point improves as the step length reduced. The maximum error indicates that the numerical result become closer to the exact solution as the step length tends to zero.

Conclusion

The Stability analysis of the hybrid variable step size Block method for integrating stiff Initial Value Problems of ODEs has been investigated. The analysis has shown that the method is zero and A-Stable capable of solving stiff initial value problem of ordinary differential equation. The numerical result obtained is found to be accurate and more efficient as the certain step size (h) tends to zero.

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